The Weak Morse Inequalities via Supersymmetry

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Goal

Our goal will be to establish the weak form of the Morse inequalities, that is $M_p \geq B_p$ where B_p is the p^{th} Betti number and M_p is the number of critical points of a Morse function h with index p.

1. The Supersymmetry Algebra

In a quantum field theory, we have a Hilbert space $\mathcal{H}$ which consists of things called states, $|a\rangle$. $\mathcal{H}$ splits as $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ where $\mathcal{H}^+$ are the "bosonic" states and $\mathcal{H}^-$ are the "fermionic" states. A supersymmetry theory contains (Hermitian) symmetry operators $Q_1, ..., Q_N$ which map $\mathcal{H}^+$ into $\mathcal{H}^-$ and vice versa.

1.1. **Properties of symmetry operators.** Let us define the operator $(-1)^F$ on $\mathcal{H}$ by $(-1)^F\psi = \psi$ for $\psi \in \mathcal{H}^+$ and $(-1)^F\chi = -\chi$ for $\chi \in \mathcal{H}^-$. The purpose of $(-1)^F$ is to distinguish $\mathcal{H}^+$ from $\mathcal{H}^-$. The symmetry operators must anticommute with $(-1)^F$:

$$(-1)^F Q_i + Q_i(-1)^F = 0,$$

in particular, they are odd operators. The Q_i must also commute with the Hamiltonian operator H which generates time translations:

$$Q_i H - H Q_i = 0.$$

Finally, to specify the algebraic structure we have to require the following

$$Q_i^2 = H, \quad \text{for all } i \tag{1}$$

$$Q_i Q_j + Q_j Q_i = 0, \quad \text{for } i \neq j. \tag{2}$$

This is the form of the supersymmetry algebra that we will use.

1.2. **Example.** Let M be an n-dimensional Riemannian manifold, and denote by Ω^p the space of p-forms on M. We interpret a p-form as bosonic or fermionic according to whether p is even or odd (for our purposes, it won't matter which one is which). Then we can define the symmetry operators as follows:

$$Q_1 = d + d^*, \quad Q_2 = i(d - d^*), \quad H = dd^* + d^*d$$

where d is the usual exterior derivative, and d^* its adjoint. So since $d^2 = (d^*)^2 = 0$ we have the supersymmetry relations (1) and (2).

1.3. **A generalization of the example.** To prove the weak Morse inequalities, we will need to make use of a simple generalization of the above example. Let $h : M \to \mathbb{R}$ be a smooth function and let $t \in \mathbb{R}$. We define

$$d_t = e^{-ht} d e^{ht} \quad \text{and} \quad d_t^* = e^{ht} d^* e^{-ht}.$$

The properties of d and d^* imply that $d_t^2 = (d_t^*)^2 = 0$, so defining

$$Q_{1t} = d_t + d_t^*, \quad Q_{2t} = i(d_t - d_t^*), \quad H_t = d_t d_t^* + d_t^* d_t \tag{3}$$

we see that (3) satisfies (1) and (2) for any t.

2. The Weak Morse Inequalities

2.1. Basic facts.

2.1.1. *Betti numbers.* We can define a Betti number $B_p(t)$ as the number of linearly independent p-forms which are d_t-closed, but not d_t-exact (i.e., the p-forms ψ such that $d_t\psi = 0$ but for which there is no form χ for which $\psi = d_t\chi$). d_t differs from d by conjugation by e^{ht}, an invertible operator. Since e^{ht} is invertible, it gives an invertible mapping $\psi \mapsto e^{ht}\psi$ which sends closed but not exact p-forms in the usual sense to d_t-closed, but not d_t-exact p-forms. Thus $B_p(t)$ is independent of t, and hence $B_p(t) = B_p(0) = B_p$, the usual p^{th} Betti number.

2.1.2. *A word on H_t.* Because we have this independence of t, we get that the number of zero eigenvalues of H_t acting on p-forms is B_p (this is what it is when $t = 0$). This independence of t is really useful because for large t, the spectrum of H_t vastly simplifies. We will use this to bound B_p above in terms of the number of critical points of h.

2.2. Critical points of h.

2.2.1. *Notation.* At each point $p \in M$, choose an orthonormal basis of tangent vectors $a^k(p)$. We can think of the $a^k(p)$ as operators on the exterior algebra via contraction. Let a^{k*} be the adjoint operators of a^k, these act by exterior multiplication by the 1-form dual to a^k.

2.2.2. *Why critical points of h?* On a Riemannian manifold, we can we can speak of the covariant second derivative of h with components $\dfrac{D^2h}{D\phi^i D\phi^j}$ in the basis dual to the a^k. We can then calculate that

$$H_t = dd^* + d^*d + t^2(dh)^2 + \sum_{i,j} t \frac{D^2h}{D\phi^i D\phi^j}[a^{*i}, a^j]$$

where $(dh)^2 = \gamma^{ij}\left(\frac{\partial h}{\partial \phi^i}\right)\left(\frac{\partial h}{\partial \phi^j}\right)$ is the square of the the gradient of h, evaluated with respect to the Riemannian metric γ on M.

This expansion shows why the critical points of h are important. The term $t^2(dh)^2$ becomes large as t becomes large, except near the critical points of h, i.e., where $dh = 0$. Thus the eigenfunctions of H_t are concentrated near the critical points of h, and an asymptotic expansion for the eigenvalues in powers of $\frac{1}{t}$ can be calculated in terms of the local data at the critical points.

2.2.3. *Calculating the spectrum of H_t.* We will consider the case of a nondegenrate Morse function h, so that it has isolated critical points q^a, and so at each of these points, the matrix $\left(\frac{D^2 h}{D\phi^i D\phi^j}\right)$ is nonsingular. Let M_p be the number of critical points with index p.

Let $\lambda_p^{(n)}(t)$ be the n^{th} smallest eigenvalue of H_t acting on p-forms. We will look for an asymptotic expansion of $\lambda_p^{(n)}(t)$ of the form:

$$\lambda_p^{(n)}(t) = t\left(A_p^{(n)} + \frac{B_p^{(n)}}{t} + \frac{C_p^{(n)}}{t^2} + \cdots\right). \tag{4}$$

As noted above, the B_p is equal to the number of $\lambda_p^{(n)}(t)$ which vanish. For large enough t, we can see that the number of $\lambda_p^{(n)}(t)$ which vanish is no larger than the number of $A_p^{(n)}$ which vanish. To establish the weak Morse inequality $M_p \geq B_p$, we will need to argue that M_p is no smaller than the number of $A_p^{(n)}$ which vanish (in fact we will see that those numbers are equal).

Let ϕ_i be local coordinates so that the critical point is the origin, and the metric tensor is Euclidean up to terms of order ϕ^2. We can choose these coordinates so that

$$h(\phi_i) = h(0) + \frac{1}{2}\sum \lambda_i \phi_i^2 + O(\phi^3)$$

for some λ_i. With these coordinates we can approximate H_t near q^a as

$$\bar{H}_t = \sum_i \left(-\frac{\partial^2}{\partial \phi_i^2} + t^2\lambda_i^2\phi_i^2 + t\lambda_i[a^{i*}, a^i]\right) \tag{5}$$

We shall calculate the spectrum of (5), which we will write as

$$\bar{H}_t = \sum (H_i + t\lambda_i K_i)$$

where

$$H_i = -\frac{\partial^2}{\partial \phi_i^2} + t^2\lambda_i^2\phi_i^2, \quad \text{and} \quad K_j = [a^{j*}, a^j].$$

The H_i and K_j commute and can be simultaneously diagonalized. The eigenvalues of H_i are $t|\lambda_i|(1+2N_i)$, $N_i \in \mathbb{N}_0$, each having multiplicity 1. The eigenvalues of K_j are ± 1. Putting this together we have the eigenvalues of H_t are

$$t\sum_i \left(|\lambda_i|(1+2N_i) + \lambda_i n_i\right), \quad N_i \in \mathbb{N}_0,\ n_i = \pm 1 \tag{6}$$

This is the spectrum of H_t. To get the eigenvalues of H_t acting on p-forms, we must require the number of positive n_i to be p.

2.3. **Conclusion.** In order for (6) to vanish, we must set all $N_i = 0$, and choose n_i to be positive if and only if λ_i is negative. So, expanding around any critical point, we see that H_t has precisely one eigenvalue of 0, and it is a zero eigenvalue for p-forms if the critical point has index p. All the other eigenvalues are positive multiples of t. (6) gives the coefficient $A_p^{(n)}$. Thus we have that the number of zero eigenvalues for p-forms is equal to the number of critical points of index p. But we already know that B_p is bounded above by the number of zero eigenvalues for p-forms, and hence

$$M_p \geq B_p.$$

References

[1] D. Freed and K. Uhlenbeck, editors, *Geometry and Quantum Field Theory*, IAS/Park City Mathematics Series, Vol. 1, 1995.

[2] R. Ticciati, *Quantum Field Theory for Mathematicians*, Encyclopedia of Mathematics, 1999.

[3] E. Witten, *Supersymmetry and Morse Theory*, J. Differential Geometry **17** (1982) 661-692.